Journal of Engineering Mathematics, Vol. 11, No. 1, January 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

Some comments on steady, laminar flow through twisted pipes

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(Received September 2, 1975)

SUMMARY
Steady laminar flow through pipes of straight centre line and fixed cross-sectional geometry is considered *for pipes in which the orientation of the cross-section changes slowly with distance along the axis of the pipe.* For small rates of twist, the (local) departure from (local) Poiseuille flow is small and it is shown that a part of the mathematical problem for this secondary flow is identical to that for the small, transverse displacement of a clamped elastic plate due to constant loading. A detailed examination of the loss of flow rate (due to the twist) is given. The special case of a pipe of elliptical cross-section is found to be analytically tractable (with the aid of a computer) and is considered in detail.

1. Introduction

Poiseuille flow in cylindrical pipes is well understood; see, for example, Batchelor [1]. In the past fifty years, there have been many investigations, both theoretical and experimental, on the effect of slowly varying changes in duct geometry. Such changes cause a loss of flow rate, for a given applied pressure difference, and can considerably affect the critical Reynold's number for transition to turbulence. Much of the present interest in the topic can be traced to bio-mechanical applications. In the latter, the flexibility of the bounding surface is an important consideration. We shall content ourselves with referring the reader to papers [2]–[8], which are recent publications in the field.

The present paper is somewhat unique in type. We consider a rigid pipe of straight centre line and fixed cross-sectional geometry, the cross-sectional geometry being the area which is inside the pipe and which lies on a plane perpendicular to the central axis. The twist of the pipe takes the form of a varying *orientation* of the cross-section with distance along the straight central axis; see Figure 1.

Manton [9] considered a case where the area (but not the orientation) of the cross-section of the pipe varied slowly with position along the straight central axis. Dean [10] considered a case where the area of the cross-section was constant but the central axis was slightly curved. The present paper presents another quite different geometrical departure from the straight, cylindrical pipe. In many real life applications, all these effects will occur at the same time. It is hoped that the present paper will give some further understanding to what is a very complex topic.

Our analysis is based on a regular perturbation expansion in terms of the appropriate small parameters. Naturally, situations in which one or more of these parameters is large are outside the scope of the theory. We deal explicitly only with cases where the central axis coincides with the geometric centre of the cross-sectional area. Should this paper promote further interest, then cases where the central axis (of twist) does not coincide with the centre of the cross-section should be examined. Indeed, there is no reason why this axis must lie inside the pipe.

We consider a fluid of constant density and viscosity. We assume that the thermal effects caused by the fluid motion are negligible. The effect of gravity is combined into the pressure term, so that our pressure, p, is in fact the "modified pressure" (as explained in Batchelor [1]).

2. The coordinate system and the full equations of motion

The coordinate system used, i.e. X, Y, Z, is illustrated in Figure 1. We define $Z \equiv z$ and use

$$\frac{\partial}{\partial Z} \equiv \text{rate of change as } z \text{ varies, } \textit{with } X \textit{ and } Y \textit{ fixed.}$$

The author examined the possibility of decomposing vector quantities into contravariant components and thus of using a metric tensor. This did not prove convenient.

Let $\hat{\boldsymbol{X}}$, $\hat{\boldsymbol{Y}}$ and $\hat{\boldsymbol{k}}$ be unit vectors in the OX, OY and Oz directions, respectively. We decompose the velocity vector, $\boldsymbol{V}$, as

$$\boldsymbol{V} = V_X\hat{\boldsymbol{X}} + V_Y\hat{\boldsymbol{Y}} + V_z\hat{\boldsymbol{k}}. \tag{1}$$

The full Navier–Stokes equations for steady flow of a fluid of constant density ρ and constant (kinematic) viscosity ν are then as given below; p is the fluid pressure.

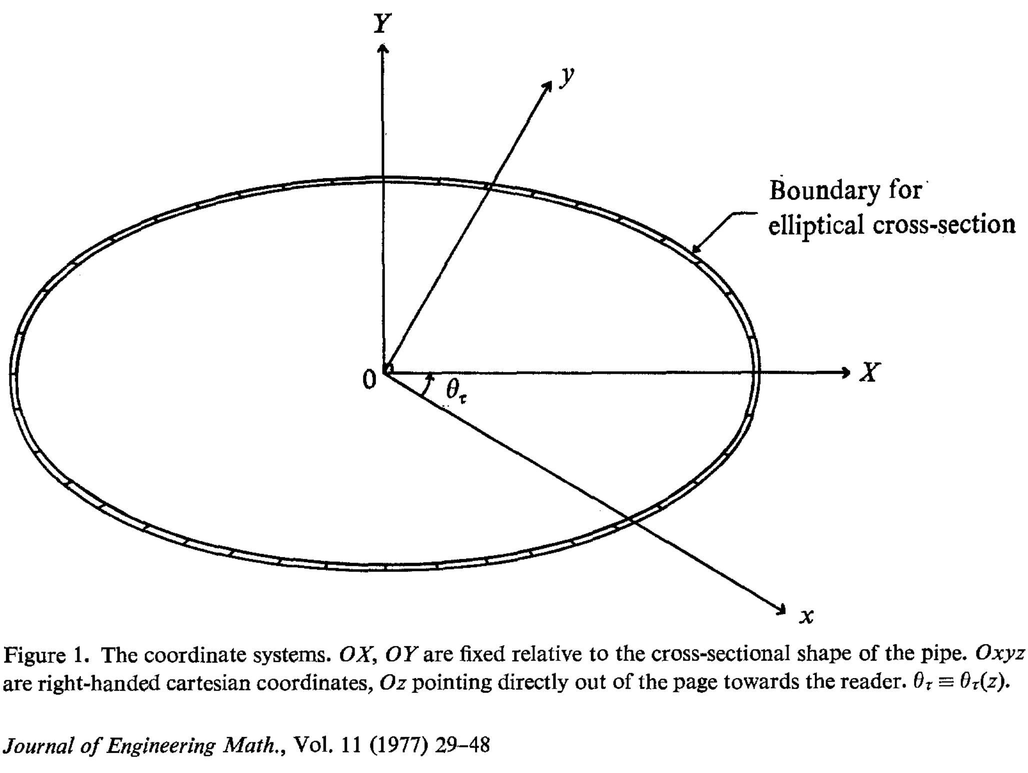

Figure 1. The coordinate systems. OX, OY are fixed relative to the cross-sectional shape of the pipe. $Oxyz$ are right-handed cartesian coordinates, Oz pointing directly out of the page towards the reader. $\theta_\tau \equiv \theta_\tau(z)$.

Continuity:

$$\frac{\partial V_X}{\partial X} + \frac{\partial V_Y}{\partial Y} + \frac{\partial V_z}{\partial Z} + \theta'_\tau \left(Y \frac{\partial V_z}{\partial X} - X \frac{\partial V_z}{\partial Y} \right) = 0, \tag{2}$$

$\hat{X}$ component of equation of motion:

$$\begin{aligned}
&\frac{1}{v} V_X \frac{\partial V_X}{\partial X} + \frac{1}{v} V_Y \frac{\partial V_X}{\partial Y} + \frac{1}{v} V_z \frac{\partial V_X}{\partial Z} + \frac{1}{v} \theta'_\tau V_z \\
&\quad \times \left\{ Y \frac{\partial V_X}{\partial X} - X \frac{\partial V_X}{\partial Y} - V_Y \right\} + \frac{1}{\rho v} \cdot \frac{\partial p}{\partial X} \\
&\quad = \nabla^2_{X,Y,Z} V_X - \theta''_\tau V_Y + 2\theta'_\tau \left\{ Y \frac{\partial^2 V_X}{\partial X \partial Z} - X \frac{\partial^2 V_X}{\partial Y \partial Z} - \frac{\partial V_Y}{\partial Z} \right\} \\
&\quad + (\theta'_\tau)^2 \left\{ Y^2 \frac{\partial^2 V_X}{\partial X^2} + X^2 \frac{\partial^2 V_X}{\partial Y^2} - 2XY \frac{\partial^2 V_X}{\partial X \partial Y} - Y \frac{\partial V_X}{\partial Y} \right. \\
&\quad \left. - X \frac{\partial V_X}{\partial X} - V_X + 2X \frac{\partial V_Y}{\partial Y} - 2Y \frac{\partial V_Y}{\partial X} \right\} + \theta''_\tau \left\{ Y \frac{\partial V_X}{\partial X} - X \frac{\partial V_X}{\partial Y} \right\},
\end{aligned} \tag{3}$$

$\hat{Y}$ component of equation of motion:

$$\begin{aligned}
&\frac{1}{v} V_X \frac{\partial V_Y}{\partial X} + \frac{1}{v} V_Y \frac{\partial V_Y}{\partial Y} + \frac{1}{v} V_z \frac{\partial V_Y}{\partial Z} + \frac{1}{v} \theta'_\tau V_z \\
&\quad \times \left\{ Y \frac{\partial V_Y}{\partial X} - X \frac{\partial V_Y}{\partial Y} + V_X \right\} + \frac{1}{\rho v} \cdot \frac{\partial p}{\partial Y} \\
&\quad = \nabla^2_{X,Y,Z} V_Y + \theta''_\tau V_X + 2\theta'_\tau \left\{ Y \frac{\partial^2 V_Y}{\partial X \partial Z} - X \frac{\partial^2 V_Y}{\partial Y \partial Z} + \frac{\partial V_X}{\partial Z} \right\} \\
&\quad + (\theta'_\tau)^2 \left\{ Y^2 \frac{\partial^2 V_Y}{\partial X^2} + X^2 \frac{\partial^2 V_Y}{\partial Y^2} - 2XY \frac{\partial^2 V_Y}{\partial X \partial Y} - Y \frac{\partial V_Y}{\partial Y} \right. \\
&\quad \left. - X \frac{\partial V_Y}{\partial X} - V_Y + 2Y \frac{\partial V_X}{\partial X} - 2X \frac{\partial V_X}{\partial Y} \right\} + \theta''_\tau \left\{ Y \frac{\partial V_Y}{\partial X} - X \frac{\partial V_Y}{\partial Y} \right\},
\end{aligned} \tag{4}$$

$\hat{k}$ component of equation of motion:

$$\begin{aligned}
&\frac{1}{v} V_X \frac{\partial V_z}{\partial X} + \frac{1}{v} V_Y \frac{\partial V_z}{\partial Y} + \frac{1}{v} V_z \frac{\partial V_z}{\partial Z} + \frac{1}{v} \theta'_\tau V_z \left(Y \frac{\partial V_z}{\partial X} - X \frac{\partial V_z}{\partial Y} \right) \\
&\quad + \frac{1}{\rho v} \left(\theta'_\tau Y \frac{\partial p}{\partial X} - \theta'_\tau X \frac{\partial p}{\partial Y} + \frac{\partial p}{\partial Z} \right) \\
&\quad = \nabla^2_{X,Y,Z} V_z + 2\theta'_\tau \left\{ Y \frac{\partial^2 V_z}{\partial X \partial Z} - X \frac{\partial^2 V_z}{\partial Y \partial Z} \right\} + \theta''_\tau \left\{ Y \frac{\partial V_z}{\partial X} - X \frac{\partial V_z}{\partial Y} \right\} \\
&\quad + (\theta'_\tau)^2 \left\{ Y^2 \frac{\partial^2 V_z}{\partial X^2} + X^2 \frac{\partial^2 V_z}{\partial Y^2} - 2XY \frac{\partial^2 V_z}{\partial X \partial Y} - Y \frac{\partial V_z}{\partial Y} - X \frac{\partial V_z}{\partial X} \right\}.
\end{aligned} \tag{5}$$

3. Steady laminar flow in pipes of zero twist

Such pipes are cylindrical and the flow is unidirectional, being known as Poiseuille flow. In fact,

$$V \equiv V_0(X,Y)\hat{k} \equiv V_0 \text{ (say)} \tag{6}$$

and

$$p \equiv -Pz + \text{constant} = p_0 \text{ (say)}, \tag{7}$$

where P is a constant (applied pressure gradient) and

$$\rho\nu\nabla^2 V_0 = -P. \tag{8}$$

We will take P to be positive. This places no real restriction. V_0 is determined uniquely by (8) and the boundary condition;

$$V_0 = 0 \text{ at the (inside) pipe walls.} \tag{9}$$

We define

$$U_0 \equiv V_0(0,0), \tag{10}$$

assuming that U_0 represents a typical value of V_0.

We define

$$L \equiv \text{typical length scale of the cross-section,} \tag{11}$$

and

$$R \equiv \left(\frac{U_0 L}{\nu}\right) \equiv \text{Reynolds number of the flow.} \tag{12}$$

Obviously,

$$U_0 \sim (PL^2/\rho\nu). \tag{13}$$

We define the flow rate

$$Q \equiv \iint\limits_{\substack{\text{cross-}\\\text{section}}} V_0(X,Y)\,dA \equiv \frac{PL^4}{\rho\nu} \text{ (non-dimensional function}^\star \text{ of } (X/L), (Y/L)). \tag{14}$$

Obviously,

$$Q \sim (PL^4/\rho\nu), \tag{15}$$

and for any given pipe shape, P is determined from Q (or vice-versa). Similarly, U_0, R and V_0 are determined if Q is specified.

★ This function is determined solely by the shape of the cross-section.

4. Steady flow in pipes of small twist

We now consider pipes of *small* twist, i.e. appreciable changes in $\theta_\tau \equiv \theta_\tau(z)$ (and its first few derivatives), take place over many pipe diameters. We are interested in laminar flow in such pipes when a constant pressure difference is applied between the ends of the pipe.

It is permissible to argue that the local flow (far from the ends) is basically the Poiseuille flow *appropriate to a fixed flow rate, Q*, provided

(i) $|RL\theta|'_\tau \ll 1$ (16)

and

(ii) $|L\theta'_\tau| \ll 1, \quad |\theta''_\tau| \ggg |\theta'_\tau|^2.$ (17)

θ'_τ, θ''_τ are, respectively, the first and second derivatives of θ_τ (with respect to z of course) and the terminology of Section 3 is used. It is assumed that

$$\frac{\partial}{\partial Z} \sim \theta'_\tau \text{ (or smaller).} \tag{18}$$

Let

$$V \equiv V_0 + (V_X\hat{X} + V_Y\hat{Y} + \tilde{V}_z\hat{k}) \tag{19}$$

and

$$p \equiv p_0 + p_1(X, Y, Z), \tag{20}$$

where is it assumed that

$$\left|\frac{\partial \tilde{V}_z}{\partial Z}\right| \ll \left(\left|\frac{\partial V_X}{\partial X}\right| + \left|\frac{\partial V_Y}{\partial Y}\right|\right) \text{ (typically)} \tag{21}$$

and

$$|\tilde{V}_z|,\ |V_X|,\ |V_Y| \text{ are everywhere much smaller than } U_0. \tag{22}$$

The inertia terms in the $\hat{X}$ and $\hat{Y}$ components of the equation of motion are negligible, as are all the terms on the right-hand sides except the first. We therefore deduce that to a first approximation

$$\frac{\partial p_1}{\partial X} = \rho\nu\nabla^2_{X,Y}V_X \tag{23}$$

and

$$\frac{\partial p_1}{\partial Y} = \rho\nu\nabla^2_{X,Y}V_Y. \tag{24}$$

Similarly, the equation of continuity reduces to

$$\frac{\partial V_X}{\partial X} + \frac{\partial V_Y}{\partial Y} = \theta'_\tau\left(X\frac{\partial V_0}{\partial Y} - Y\frac{\partial V_0}{\partial X}\right). \tag{25}$$

We have shown that, to a first approximation, V_X and V_Y are obtained by solving the (essentially) two-dimensional problem set out by equations (23), (24) and (25) together with the boundary conditions:

$$V_X = 0 = V_Y \text{ at the (inside) wall.} \tag{26}$$

We now proceed to reformulate this two-dimensional problem in a way which is preferable. Let $\tilde{V}_X$, $\tilde{V}_Y$ denote the first approximation to V_X, V_Y, respectively, as obtained from the solutions of (23) to (26).

Let $\hat{\boldsymbol{Z}}$ be a unit vector along a curve on which X, Y is fixed and z varies; this unit vector pointing in the direction of increasing z. A fluid particle with velocity $\equiv$ (scalar function) $\hat{\boldsymbol{Z}}$, would not change its position relative to the cross-section of the pipe. A careful examination reveals that it is the velocity difference

$$\boldsymbol{V} - \{V_z/(\hat{\boldsymbol{Z}}\cdot\hat{\boldsymbol{k}})\}\hat{\boldsymbol{Z}}$$

which will cause migration of a fluid particle *relative to the pipe cross-section.* Now

$$\hat{\boldsymbol{Z}} \equiv (\hat{\boldsymbol{k}} - \theta_\tau' Y\hat{\boldsymbol{X}} + \theta_\tau' X\hat{\boldsymbol{Y}})/[1 + (\theta_\tau')^2(X^2 + Y^2)]^{\frac{1}{2}}. \tag{27}$$

Thus the migration is governed, to a first approximation, by the velocity

$$(\tilde{V}_X + \theta_\tau' YV_0)\hat{\boldsymbol{X}} + (\tilde{V}_Y - \theta_\tau' XV_0)\hat{\boldsymbol{Y}}. \tag{28}$$

We define

$$V_X^* = \tilde{V}_X + \theta_\tau' YV_0 \tag{29}$$

and

$$V_Y^* = \tilde{V}_Y - \theta_\tau' XV_0. \tag{30}$$

(23)–(26) require that

$$\frac{\partial p_1}{\partial X} = \rho\nu\nabla_{X,Y}^2 V_X^* - 2\rho\nu\theta_\tau' \frac{\partial V_0}{\partial Y} + \theta_\tau' YP, \tag{31}$$

$$\frac{\partial p_1}{\partial Y} = \rho\nu\nabla_{X,Y}^2 V_Y^* + 2\rho\nu\theta_\tau' \frac{\partial V_0}{\partial X} - \theta_\tau' XP, \tag{32}$$

$$\frac{\partial V_X^*}{\partial X} + \frac{\partial V_Y^*}{\partial Y} = 0, \tag{33}$$

and the boundary conditions

$$V_X^* = V_Y^* = 0 \text{ at the (inside) wall of the pipe.} \tag{34}$$

Equation (8) was used in the derivation of (31) and (32).

We introduce the stream function Ψ^*, where

$$V_X^* \equiv \frac{\partial \Psi^*}{\partial Y}, \quad V_Y^* \equiv -\frac{\partial \Psi^*}{\partial X} \tag{35}$$

and $\Psi^* = 0$ on the boundary.

We eliminate p_1 between (31) and (32) and use (8) and (35) to obtain

$$\rho\nu\nabla^4_{X,Y}\Psi^* = -4\theta'_\tau P, \tag{36}$$

where we require

$$\frac{\partial\Psi^*}{\partial n} = 0 = \Psi^* \text{ at the boundary.} \tag{37}$$

$\partial/\partial n$ is the (partial) derivative in the direction of the inward normal to the boundary.

Equations (36) and (37) are those governing the small, transverse displacement of a clamped, elastic plate *of the shape of our pipe cross-section*; see, for example, Love [11]. Thus all the results of this part of classical elasticity theory have an interpretation in fluid mechanics. We should also note that (35), (36) and (37) have another analogy in plane, Stokes flow. They represent such a flow, in a region identical to the cross-section of the pipe, activated by a uniform distribution of vorticity sources. The problem can be reformulated in a number of ways by noting that the general solution of (36) is

$$\Psi^* = -\theta'_\tau(X^2 + Y^2)V_0 + (\text{any biharmonic function}),$$

where V_0 is *any* solution of equation (8).

As the reader will be aware, simple closed-form solutions of (36) and (37) are not over-abundant and recourse must often be made to numerical procedures. In Section 5, we discuss in detail the special case of an elliptic cross-section. Arguments similar to those given in Section 5 can be made in the general case and lead to the following conclusions.

(i) V_X and V_Y (as well as V_X^*, V_Y^*) are $O((\theta'_\tau L)U_0)$.

(ii) $\tilde{V}_z$ is $O((R\theta'_\tau L)U_0, (\theta'_\tau L)^2U_0)$ provided that $J(V_0,\Psi^*/X, Y) \not\equiv 0$. The relative increase in applied pressure gradient (over that for the Poiseuille flow) needed to maintain the flow rate Q is $O((R\theta'_\tau L), (\theta'_\tau L)^2)$.

(iii) If $J(V_0,\Psi^*/X, Y) \equiv 0$ (as happens in the case of the elliptical cross-section) then $\tilde{V}_z$ is $O((R\theta'_\tau L)^2U_0, (\theta'_\tau L)^2U_0)$, and the required relative increase in the applied pressure gradient is $O((R\theta'_\tau L)^2, (\theta'_\tau L)^2)$.

In fact, physical considerations would lead us to believe that in the general case,

$$0 \not\equiv \zeta \equiv \frac{\left|J\left(\dfrac{V_0,\Psi^*}{X, Y}\right)\right|}{\|\nabla V_0\|\;\|\nabla\Psi^*\|} \ll 1. \tag{38}$$

This is because the contours of V_0 are not identical to, but are very similar to those for Ψ^*. It is therefore probably useful to obtain (in the general case) the $(R\theta'_\tau L)^2$ contribution to $(\tilde{V}_z/U_0)$ and to the pressure gradient, as well as the $(R\theta'_\tau L)$ contribution. This can be done in very much the same way as the $(R\theta'_\tau L)^2$ contribution is obtained in Section 5 for the elliptical case.

The result (38) was shown to hold in the case of a square cross-section. Although there is no closed-form analytic solution for a square cross-section, the numerical solution was available to sufficient accuracy for the present purposes; we mention here the work of Wojtaszak [13] and Timoshenko and Woinowsky–Krieger [12]. An analytic solution was available for V_0 and this presented no computational problems. We found that ζ was at most, about .046, a typical value being about .010. This strengthens the point made in the proceeding paragraph.

(iv) The results (i), (ii) and (iii) show that $|\partial \tilde{V}_z/\partial Z|$ is typically

$$O\left((R\theta_\tau' L)\left(\left|\frac{\partial V_X}{\partial X}\right| + \left|\frac{\partial V_Y}{\partial Y}\right|\right)\right)$$

and thus the results are consistent with the assumption (21). We note that $|\tilde{V}_z|$ can be much larger than $|V_X|$ and $|V_Y|$ if $R \gg 1$.

5. Steady flow in a cylindrical pipe of elliptic cross-section in the presence of small twist

We choose OX, OY to coincide with the major, minor axes of the ellipse, respectively, as shown in Figure 1. The (inner) boundary of the pipe is given by

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \tag{39}$$

where a, b are positive constants; $a > b$ and b is not *much* smaller than a.

In this case,

$$V_0 = \frac{P}{2\rho\nu}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-1}\left(1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2}\right) = U_0\left(1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2}\right), \tag{40}$$

$$P = \frac{4\rho\nu}{\pi}\cdot\frac{(a^2 + b^2)}{a^3b^3}\,Q, \qquad L = a \tag{41}$$

and

$$U_0 = \frac{2}{\pi ab}\,Q = \frac{P}{2\rho\nu}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-1}. \tag{42}$$

For convenience, we define

$$\eta = (b/a), \qquad \mathscr{X} = (X/a) \text{ and } \mathscr{Y} = (Y/b). \tag{43}$$

For this special case, equations (23) to (26) give

$$\tilde{V}_X = (1 - \eta)(\theta_\tau' a)\mathscr{Y} V_0 \alpha_{01} \tag{44}$$

and

$$\tilde{V}_Y = (1 - \eta)(\theta_\tau' a)\mathscr{X} V_0 \alpha_{10}, \tag{45}$$

where α_{10}, α_{01} are constants given by

$$\alpha_{01} = \eta\frac{(1 + \eta)(1 + 3\eta^2)}{(3 + 2\eta^2 + 3\eta^4)} = \eta\frac{(1 + 3\eta^2)}{(3 + \eta^2)}\alpha_{10}. \tag{46}$$

Table 1 lists the values of these two constants for various values of η. We see from these results that

$$|V_X| \sim |V_Y| \sim (1 - \eta)|\theta_\tau' a|\, U_0 \ll U_0$$

TABLE 1

Some numerical values of certain coefficients which are rational functions of η.
The values to be multiplied by ten to the power of the associated integer, e.g. $\alpha_{02} = (.174)10^{-2}$ *for* $\eta = .05$; $\alpha_{20} = (-2.078)10^{0} = -2.078$ *for* $\eta = .05$.

η	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
α_{01}	−1	−1	−1	−1																
	.176	.375	.604	.871	.118	.155	.196	.244	.297	.356	.419	.486	.555	.625	.695	.763	.829	.890	.948	1.000
α_{10}	1.049	1.096	1.141	1.183	1.220	1.254	1.281	1.302	1.316	1.322	1.319	1.308	1.289	1.262	1.229	1.189	1.146	1.099	1.050	1.000
α_{2p}	−1																			
	.527	.112	.179	.255	.341	.438	.546	.665	.792	.925	1.063	1.201	1.336	1.465	1.585	1.694	1.791	1.874	1.944	2.000
α_s	−1																			
	.667	.134	.201	.270	.339	.408	.478	.546	.614	.678	.739	.794	.844	.888	.924	.953	.974	.989	.997	1.000
α_{1p}	.368	.405	.444	.485	.529	.575	.622	.669	.717	.763	.807	.847	.884	.916	.943	.965	.981	.992	.998	1.000
α_{3p}	−20	−19	−19	−21	−19	−18	−19	−19	−19	−18	−18	−18	−19	−18	−18	−18	−18	−17	−17	−18
	−.115	−.202	.135	−.725	.745	.139	.874	−.805	−.102	.157	−.198	−.282	.785	.313	−.288	−.381	.911	.222	.262	−.605
α_{20}	−2.078	−2.110	−2.101	−2.060	−1.998	−1.924	−1.845	−1.765	−1.687	−1.612	−1.540	−1.471	−1.405	−1.341	−1.279	−1.219	−1.161	−1.105	−1.051	−1.000
α_{02}	−2	−2	−1	−1	−1	−1	−1													
	.174	.722	.166	.304	.484	.712	.990	.132	.172	.217	.269	.328	.393	.465	.543	.626	.714	.806	.901	1.000
α_{00}																−1	−1	−1	−1	
	.346	.350	.347	.338	.325	.309	.291	.272	.253	.233	.212	.191	.169	.146	.123	.988	.745	.499	.250	0.000
α_{11}	−1	−1	−1	−1																
	.176	.373	.596	.849	.114	.146	.182	.221	.264	.308	.354	.400	.445	.488	.528	.565	.597	.625	.648	.667

which is consistent with our prior assumption, (22).

Let

$$\tilde{p}_1 = -(1 - \eta)Pa(\theta_\tau' a)\mathscr{X}\mathscr{Y}\alpha_{2p}, \tag{47}$$

where

$$\alpha_{2p} = \{\eta(1 + \eta)(1 + 3\eta^2)(\eta^2 + 3)\}/\{(1 + \eta^2)(3 + 2\eta^2 + 3\eta^4)\}. \tag{48}$$

Values of α_{2p} are given in Table 1. The solution of (23) to (26) reveals that

$$p_1 = f(Z) + \tilde{p}_1 + (\text{contributions from neglected terms}), \tag{49}$$

where $f(Z)$ is any function of Z. In fact, $f(Z)$ will be determined from the problem for $\tilde{V}_z$.

We now turn to the alternative formulation as given by (35), (36) and (37). The solution of these in the present special case is

$$\begin{aligned} \Psi^* &\equiv \frac{-Pa^3}{2\rho\nu}(\theta_\tau' a)\frac{\eta^4}{(3 + 2\eta^2 + 3\eta^4)}\left(1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2}\right)^2 \\ &\equiv -\frac{2\rho\nu(\theta_\tau' a)}{Pa}\cdot\frac{(1 + \eta^2)^2}{(3 + 2\eta^2 + 3\eta^4)}V_0^2. \end{aligned} \tag{50}$$

Clearly, Ψ^* is a function of V_0 and so

$$J\left(\frac{\Psi^*, V_0}{X, Y}\right) \equiv 0. \tag{51}$$

Of course, equations (44) and (45) together with (35), (29) and (30) can be shown to give the result (50).

(i) *Migration of fluid particles across the pipe*

In cartesian coordinates, the particle paths (i.e. stream lines) are given by

$$\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_0 + \tilde{V}_z};$$

where $V = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$.

We change this into our (X, Y, z) reference system and obtain

$$\frac{dX}{V_X + Y\theta_\tau'(V_0 + \tilde{V}_z)} = \frac{dY}{V_Y - X\theta_\tau'(V_0 + \tilde{V}_z)} = \frac{dz}{(V_0 + \tilde{V}_z)}. \tag{52}$$

We use (22), (29), (30) and the fact that to a first approximation $V_X \simeq \tilde{V}_X$, $V_Y \simeq \tilde{V}_Y$, to deduce that, to a first approximation, the streamlines satisfy

$$\frac{dX}{V_X^*} = \frac{dY}{V_Y^*} = \frac{dz}{V_0}. \tag{53}$$

Eq. (53) immediately yields the result (50), i.e. fluid particles move along surfaces which are geometrically similar to the boundary shape (and which are the equipotentials for

$V_0(X, Y)$). It is of interest to find out how a fluid element changes position relative to the local cross-section of the pipe. Consider a particle which flows along

$$b^2X^2 + a^2Y^2 = \alpha^2, \quad (\alpha^2 < a^2b^2). \tag{54}$$

We introduce the *eccentric* angle γ by

$$X = (\alpha/b)\cos\gamma; \quad Y = (\alpha/a)\sin\gamma. \tag{55}$$

It can be deduced from (53) that for a fluid particle

$$\gamma = -\alpha_s\theta_\tau + (\text{constant}) \leftrightarrows \frac{d\gamma}{dz} = -\alpha_s\theta'_\tau, \tag{56}$$

where

$$\alpha_s = \frac{4\eta(1+\eta^2)}{(3+2\eta^2+3\eta^4)}. \tag{57}$$

Values of α_s are given in Table 1.

For a circular pipe, $\eta = 1$. In this case, equation (57) gives $\alpha_s = 1$, i.e. $\gamma = -\theta_\tau +$ + (constant). This indicates that the fluid element is staying fixed relative to the cartesian axes Oxy, a result which is obviously correct (for a circular pipe, $V_x \equiv 0 \equiv V_y \equiv \tilde{V}_z$).

For the other cases $0 < \eta < 1$ and equation (57) shows that $0 < \alpha_s < 1$. From this fact and equation (56), we deduce that fluid particles migrate *relative to the (local) cross-section*, along an elliptical curve geometrically similar to that of the boundary shape and in the direction opposite to the local twisting of the pipe. In *real terms*, the fluid particle migrates in the direction of local twist but always lags behind (except where $\theta_\tau = 0$).

(ii) *The problem for* $\tilde{V}_z$

Equation (5), together with appropriate boundary conditions will determine $\tilde{V}_z$. We start by assuming that $\nabla^2_{X,Y}\tilde{V}_z$ will be important in the determination of $\tilde{V}_z$. If we then omit from equation (5) those terms which are definitely negligible (whatever the result for $\tilde{V}_z$), we obtain

$$\begin{aligned}
&\frac{1}{\nu}V_X\frac{\partial V_0}{\partial X} + \frac{1}{\nu}V_Y\frac{\partial V_0}{\partial Y} + \frac{1}{\nu}\theta'_\tau V_0\left(Y\frac{\partial V_0}{\partial X} - X\frac{\partial V_0}{\partial Y}\right)\\
&\quad + \frac{1}{\rho\nu}\left(\theta'_\tau Y\frac{\partial p_1}{\partial X} - \theta'_\tau X\frac{\partial p_1}{\partial Y} + \frac{\partial p_1}{\partial Z}\right)\\
&\quad = \nabla^2_{X,Y}\tilde{V}_z + (\theta'_\tau)^2\left\{Y^2\frac{\partial^2 V_0}{\partial X^2} + X^2\frac{\partial^2 V_0}{\partial Y^2} - Y\frac{\partial V_0}{\partial Y} - X\frac{\partial V_0}{\partial X}\right\}\\
&\quad + \theta''_\tau\left\{Y\frac{\partial V_0}{\partial X} - X\frac{\partial V_0}{\partial Y}\right\}.
\end{aligned} \tag{58}$$

Now the last two terms on the R.H.S. of (58) are of order $(\theta')^2U_0(1-\eta)$.

The last term on the left hand side of (58) is $O((1-\eta)U_0(\theta'_\tau)^2 + (1/\rho\nu)\,|f'(Z)|)$. $|f'(Z)|$ will be of order consistent with the rest of the terms in the equation, i.e. we can ignore it during the present discussion of order of magnitude.

At first sight, it would seem that replacing V_X, V_Y in (58) by $\tilde{V}_X$, $\tilde{V}_Y$ would give a first approximation to the sum of the first three terms on the L.H.S. and that this would be of order $(1-\eta)R\theta'_\tau U_0 a^{-1}$. But

$$\tilde{V}_X \frac{\partial V_0}{\partial X} + \tilde{V}_Y \frac{\partial V_0}{\partial Y} + \theta'_\tau V_0\left(Y \frac{\partial V_0}{\partial X} - X \frac{\partial V_0}{\partial Y}\right) \equiv \left(V_X^* \frac{\partial V_0}{\partial X} + V_Y^* \frac{\partial V_0}{\partial Y}\right)$$
$$\equiv J\left(\frac{V_0, \Psi^*}{X, Y}\right) \equiv 0, \tag{59}$$

as is shown by equation (51). This seemingly convenient cancellation does, in fact, considerably complicate the analysis. If the Jacobian of equation (59) *had* been of the same order of magnitude as the separate terms, we would have deduced that

$$\{\tilde{V}_z/(1-\eta)U_0\} \text{ was } O\{R(\theta'_\tau a), (\theta'_\tau a)^2\}. \tag{60}$$

The above cancellation reduces the $R(\theta'_\tau a)$ to something smaller. We must calculate that part of this smaller quantity which, in magnitude, could be bigger than, or comparable with, $(\theta'_\tau a)^2$.

A closer examination of the full problem enables us to deduce that

$$\begin{aligned} V_X &= \tilde{V}_X\{1 + O(R(a\theta'_\tau), (a\theta'_\tau)^2)\} + O((a\theta'_\tau)\tilde{V}_z), \\ V_Y &= \tilde{V}_Y\{1 + O(R(a\theta'_\tau), (a\theta'_\tau)^2)\} + O((a\theta'_\tau)\tilde{V}_z). \end{aligned} \tag{61}$$

The term involving $R(a\theta'_\tau)$ will generate a contribution of order $(1-\eta)U_0(Ra\theta'_\tau)^2$ to $\tilde{V}_z$ and to $(\rho v)^{-1}f'(Z)$. This contribution may be significant or even dominant. The other terms in (61) yield contributions which are relatively insignificant. Thus, in order to obtain $\tilde{V}_z$ and $f'(Z)$ to a first approximation, *valid* for all possible values of R, we must obtain the terms in $R(a\theta'_\tau)$ as set out in equation (61).

Let $\tilde{\tilde{V}}_X$ and $\tilde{\tilde{V}}_Y$ be the desired corrections to $\tilde{V}_X$ and $\tilde{V}_Y$. (This means $\tilde{\tilde{V}}_X$ and $\tilde{\tilde{V}}_Y$ are of order $\tilde{V}_X R(a\theta'_\tau)$). The problem for $\tilde{\tilde{V}}_X$ and $\tilde{\tilde{V}}_Y$ reduces to the following

$$\frac{\partial \tilde{\tilde{V}}_X}{\partial X} + \frac{\partial \tilde{\tilde{V}}_Y}{\partial Y} = 0, \tag{62}$$

$$\frac{1}{v}\tilde{V}_X \frac{\partial \tilde{V}_X}{\partial X} + \frac{1}{v}\tilde{V}_Y \frac{\partial \tilde{V}_X}{\partial Y} + \frac{1}{v}V_0 \frac{\partial \tilde{V}_X}{\partial Z} + \frac{1}{v}\theta'_\tau V_0\left\{Y \frac{\partial \tilde{V}_X}{\partial X} - X \frac{\partial \tilde{V}_X}{\partial Y} - \tilde{V}_Y\right\}$$
$$+ \frac{1}{\rho v}\frac{\partial}{\partial X}(p_1 - \tilde{p}_1) = V^2_{X,Y}\tilde{\tilde{V}}_X, \tag{63}$$

$$\frac{1}{v}\tilde{V}_X \frac{\partial}{\partial X}\tilde{V}_Y + \frac{1}{v}\tilde{V}_Y \frac{\partial \tilde{V}_Y}{\partial Y} + \frac{1}{v}V_0 \frac{\partial \tilde{V}_Y}{\partial Z} + \frac{1}{v}\theta'_\tau V_0\left\{Y \frac{\partial \tilde{V}_Y}{\partial X} - X \frac{\partial \tilde{V}_Y}{\partial Y} + \tilde{V}_X\right\}$$
$$+ \frac{1}{\rho v}\frac{\partial}{\partial Y}(p_1 - \tilde{p}_1) = V^2_{X,Y}\tilde{\tilde{V}}_Y. \tag{64}$$

The boundary conditions are, of course,

$$\tilde{\tilde{V}}_X = 0 = \tilde{\tilde{V}}_Y \text{ on } \mathscr{X}^2 + \mathscr{Y}^2 = 1 = \left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2. \tag{65}$$

The solution to the above (essentially) two-dimensional problem is of the form

$$\tilde{\tilde{V}}_X = \{\text{Polynomial in } (\mathscr{X}, \mathscr{Y})\} R(\theta'_\tau a)^2 (1 - \eta) U_0 + \{\text{Polynomial in } (\mathscr{X}, \mathscr{Y})\} R\theta''_\tau a^2 (1 - \eta) U_0 \tag{66}$$

with $\tilde{\tilde{V}}_Y$ given by a similar expression. We also deduce that

$$p_1 - \tilde{p}_1 = \tilde{f}(Z) + \mathscr{X}^2\{\text{Polynomial in } (\mathscr{X}, \mathscr{Y})\} R(\theta'_\tau a)^2 (1 - \eta) Pa + \mathscr{X}\mathscr{Y}\{\text{Polynomial in } (\mathscr{X}, \mathscr{Y})\} R\theta''_\tau a^2 (1 - \eta) Pa + \text{smaller contributions due to neglected terms.} \tag{67}$$

The coefficients* in the polynomials are complicated rational functions of η; bounded over the range of interest $0 < \eta < 1$. $\tilde{f}(Z)$ is an unknown function of $z \equiv Z$ which will be determined from the problem for $\tilde{V}_z$. Equation (67) is consistent with the earlier equation (49). $\tilde{f}(Z)$ differs from $f(Z)$ in that it includes a term of order $R^2(a\theta'_\tau)Pa(1 - \eta)$ which was previously among the neglected terms.

We are now ready to solve for $\tilde{V}_z$ and $\tilde{f}(Z)$. Now the second and third term (as well as the last term) on the R.H.S. of (67) do not provide significant contributions to $\tilde{V}_z$ and $\tilde{f}(Z)$ and thus we solve

$$\frac{1}{\nu}\tilde{\tilde{V}}_X \frac{\partial V_0}{\partial X} + \frac{1}{\nu}\tilde{\tilde{V}}_Y \frac{\partial V_0}{\partial Y} + \frac{1}{\rho\nu}\left(\theta'_\tau Y \frac{\partial \tilde{p}_1}{\partial X} - \theta'_\tau X \frac{\partial \tilde{p}_1}{\partial Y} + \frac{\partial \tilde{p}_1}{\partial Z}\right) + \frac{1}{\rho\nu}\frac{d\tilde{f}}{dZ}$$
$$= \nabla^2_{X,Y}\tilde{V}_z + (\theta'_\tau)^2\left\{Y^2 \frac{\partial^2 V_0}{\partial X^2} + X^2 \frac{\partial^2 V_0}{\partial Y^2} - Y\frac{\partial V_0}{\partial Y} - X\frac{\partial V_0}{\partial X}\right\}$$
$$+ \theta''_\tau\left\{Y\frac{\partial V_0}{\partial X} - X\frac{\partial V_0}{\partial Y}\right\}. \tag{68}$$

The boundary conditions are

$$\tilde{V}_z = 0 \text{ on } a^2X^2 + b^2Y^2 = a^2b^2 \tag{69}$$

and

$$\iint_A \tilde{V}_z dA = 0, \tag{70}$$

where A is any cross-section of the pipe, i.e. A is the area

$$z = \text{constant};\ b^2X^2 + a^2Y^2 < a^2b^2.$$

The boundary condition (70) comes from the fact that

$$\iint_A (V_0 + \tilde{V}_z)dA = Q \equiv \text{flow rate},$$

and that *by definition*

$$\iint_A V_0 dA = Q.$$

* These coefficients were calculated numerically for various values of η but they are not presented here as, in themselves, they are not of primary interest.

The solution set is once again explicitly obtained, as it basically involves only polynomials in $\mathscr{X}$ and $\mathscr{Y}$. The solution set is

$$\begin{aligned}\tilde{V}_z = {} & (1-\eta)V_0\{(\theta_\tau' a)^2(\alpha_{20}\mathscr{X}^2+\alpha_{02}\mathscr{Y}^2+\alpha_{00})+\theta_\tau'' a^2\mathscr{X}\mathscr{Y}(\alpha_{11}-\eta(1+\eta)(1+\eta^2)^{-2}\} \\ & +(1-\eta)U_0(R\theta_\tau' a)^2\{e_{00}+e_{20}(\mathscr{X}^2-\eta^2\mathscr{Y}^2)+e_{40}\mathscr{X}^4+e_{22}\mathscr{X}^2\mathscr{Y}^2+e_{04}\mathscr{Y}^4 \\ & +e_{60}\mathscr{X}^6+e_{42}\mathscr{X}^4\mathscr{Y}^2+e_{24}\mathscr{X}^2\mathscr{Y}^4+e_{06}\mathscr{Y}^6+e_{80}\mathscr{X}^8 \\ & +e_{62}\mathscr{X}^6\mathscr{Y}^2+e_{44}\mathscr{X}^4\mathscr{Y}^4+e_{26}\mathscr{X}^2\mathscr{Y}^6+e_{08}\mathscr{Y}^8+e_{10}\mathscr{X}^{10} \\ & +e_{82}\mathscr{X}^8\mathscr{Y}^2+e_{64}\mathscr{X}^6\mathscr{Y}^4+e_{46}\mathscr{X}^4\mathscr{Y}^6+e_{28}\mathscr{X}^2\mathscr{Y}^8+e_{01}\mathscr{Y}^{10}\} \\ & +(1-\eta)U_0R^2(\theta_\tau'' a^2)\{e_{11}\mathscr{X}\mathscr{Y}+e_{31}\mathscr{X}^3\mathscr{Y}+e_{13}\mathscr{X}\mathscr{Y}^3+e_{51}\mathscr{X}^5\mathscr{Y} \\ & +e_{33}\mathscr{X}^3\mathscr{Y}^3+e_{15}\mathscr{X}\mathscr{Y}^5+e_{71}\mathscr{X}^7\mathscr{Y}+e_{53}\mathscr{X}^5\mathscr{Y}^3+e_{35}\mathscr{X}^3\mathscr{Y}^5 \\ & +e_{17}\mathscr{X}\mathscr{Y}^7+e_{91}\mathscr{X}^9\mathscr{Y}+e_{73}\mathscr{X}^7\mathscr{Y}^3+e_{55}\mathscr{X}^5\mathscr{Y}^5+e_{37}\mathscr{X}^3\mathscr{Y}^7 \\ & +e_{19}\mathscr{X}\mathscr{Y}^9\} \\ & +(1-\eta)V_0(R\theta_\tau' a)^2\alpha_{3p}\end{aligned} \tag{71}$$

and

$$\frac{d\tilde{f}}{dZ} = -P(1-\eta)\{(1-\eta)\alpha_{1p}(\theta_\tau' a)^2+\alpha_{3p}(R\theta_\tau' a)^2\} \tag{72}$$

where the coefficients α_{20}, etc. are rational functions of η. The coefficients are much too complicated for it to be a worthwhile exercise obtaining the algebraic formula for each of them. Instead, the author used a computer to derive them numerically for various values of η. Even this involved considerable work, viz.

(i) The coefficients involved in $\tilde{\tilde{V}}_X$ and $\tilde{\tilde{V}}_Y$ (as per equation (66)) were found numerically (for various values of η) by solving a set of seven simultaneous linear equations in seven unknowns and a similar set of eight equations in eight unknowns.

(ii) Based on the results of (i) above, it was then possible to obtain numerically (for the same set of values of η), the coefficients involved in (71) and (72). This required solving a set of twenty simultaneous linear equations in twenty unknowns, a similar set of fifteen equations in fifteen unknowns, and finally calculating the coefficients of $d\tilde{f}/dZ$ which can be expressed in terms of the other coefficients using the boundary condition (70). The coefficients α_{1p}, α_{2p}, α_{20}, α_{02}, α_{00} and α_{11} are given in Table 1. The coefficients $e_{00}, e_{20}, \ldots, e_{28}$ and e_{01} are given in Table 3. The coefficients $e_{11}, e_{31}, \ldots, e_{37}$ and e_{19} are given in Table 2.

Equations (7), (20), (67) and (72) allow us to deduce that

$$\begin{aligned}\frac{\partial p}{\partial z} = {} & -P\{1+(1-\eta)^2\alpha_{1p}(\theta_\tau' a)^2+(1-\eta)\alpha_{2p}\mathscr{X}\mathscr{Y}(\theta_\tau'' a^2)+(1-\eta)\alpha_{3p}(R\theta_\tau' a)^2\} \\ & +(\text{terms of smaller order}).\end{aligned} \tag{73}$$

Equation (73) indicates the increase in pressure gradient required in order to maintain the same flow rate as for Poiseuille flow, i.e. the non-twisted case. The pressure gradient for Poiseuille flow is, of course, $-P$.

The coefficient α_{3p} *was found to be zero for all values of* η. More precisely, we worked in double precision to the limit of accuracy of the computer. In so doing, we obtained results for α_{3p} of order 10^{-17}, 10^{-18} (see Table 1). This can (could) be accounted for by unavoidable round-off error. Considerable cross-checking and care went into the calculation of α_{3p}, this work spanning a period of 12 months. The result for α_{3p} is of considerable importance because it means that the first (bracketed) term on the R.H.S. of equation (73) does not give a valid first approximation, for all values of R, to the required *increase* in pressure gradient due to the twist. A careful reexamination of the full problem revealed that with $\alpha_{3p} = 0$,

$$\frac{\partial p}{\partial z} = -P\{1 + (1-\eta)^2\alpha_{1p}(\theta_\tau' a)^2 + (1-\eta)\alpha_{2p}\mathscr{X}\mathscr{Y}(\theta_\tau'' a^2)$$
$$+ (1-\eta)\text{ (a term of order } (R\theta_\tau' a)^3)\} + \text{terms of smaller order.} \tag{74}$$

However, the work involved in an explicit derivation of (74) would be prohibitive (and is hereby bequeathed to someone else). In any case, as was pointed out in Section 4, for shapes other than the ellipse, one can expect a (non-zero) contribution of order $(R\theta_\tau' a)$ to the bracketed term in (74).

We finish by pointing out that equation (71) *does* give a first approximation to $\tilde{V}_z$ for all possible values of R, $(\theta_\tau' a)$ and $\theta_\tau'' a^2$. In Figures 2, 3 and 4, we give some details of $\tilde{V}_z$ for various possible cases. These were obtained by computer calculations. We point out that for $R \gg 1$, $|\tilde{V}_z|$ may be typically large compared to $|V_X|$ and $|V_Y|$.

Calculations were also made for $R \ll 1$, $\theta_\tau' =$ constant and $\eta = 0.05(.05).95$. These revealed that the corresponding δV_z is negative everywhere for η less than about .45 and that

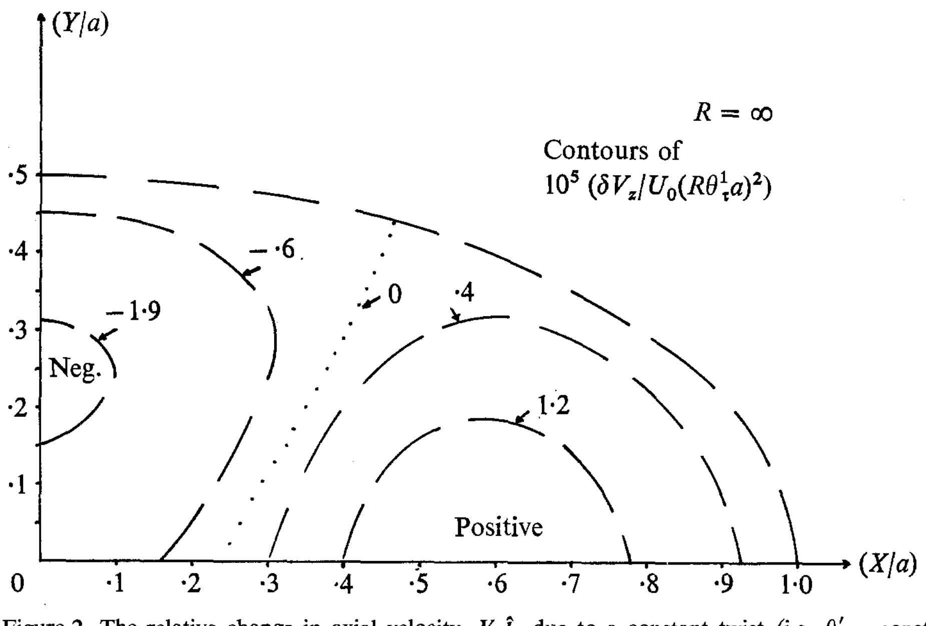

Figure 2. The relative change in axial velocity, $V\cdot\hat{k}$, due to a constant twist (i.e. $\theta_\tau' =$ constant) when $R = \infty$, $\eta = .5$ (δV_z is the excess of V_z over that which would be occasioned in the no-twist case by the same pressure gradient, i.e. $\partial p/\partial z = -P\{1 + (1-\eta)^2\alpha_{1p}(\theta_\tau' a)^2\}$. Here, V_z is taken to be $U_0 + \tilde{V}_z$.) The results for $\eta = .5$, $(R/150)^2 \gg 1$ are similar.

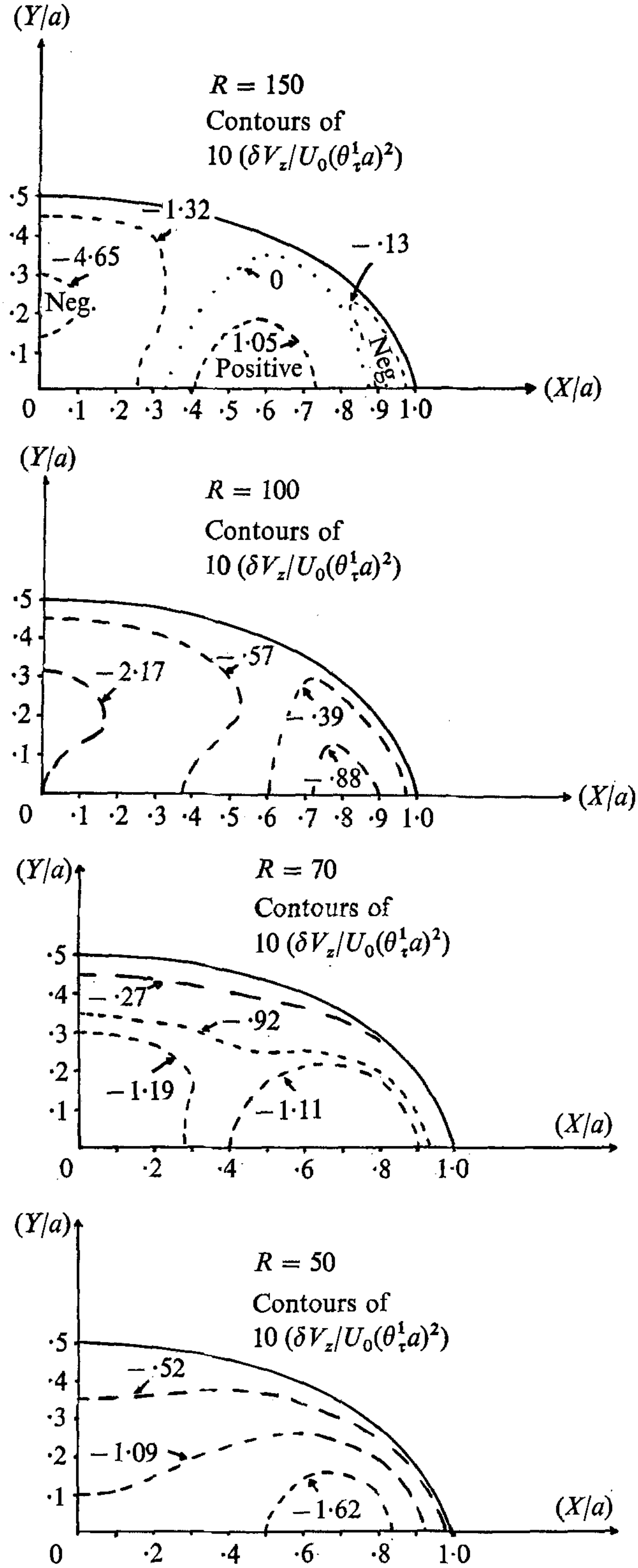

Figure 3. The relative change in axial velocity, $V \cdot \hat{k}$, due to a constant twist when $\eta = .5$ (see caption for figure 2). (a) $R = 150$, (b) $R = 100$, (c) $R = 70$, (d) $R = 50$.

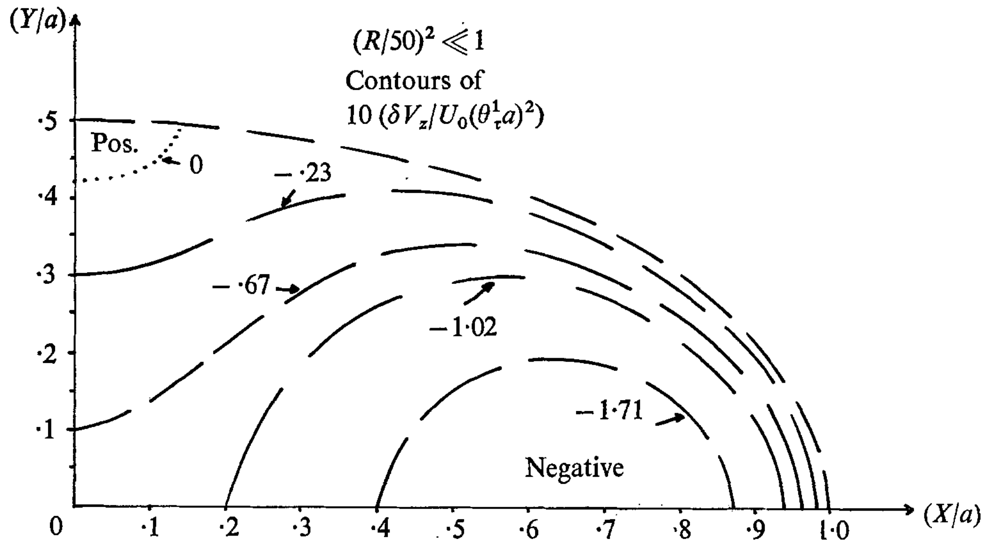

Figure 4. The relative change in axial velocity, $V \cdot \hat{k}$, due to a constant twist when $R \ll 1$, $\eta = .5$. The results for $\eta = .5$, $(R/50)^2 \ll 1$ are similar (see caption for figure 2).

as η is increased from .5, the area of positive δV_z (see top left-hand corner of Figure 4) increases in size until at $\eta = .95$, it occupies a sizable part of the cross-section. Calculations were made for $R \gg 1$, $\theta_\tau' =$ constant, and $\eta = 0.05(.05).95$. These were all qualitatively the same as that illustrated in Figure 2 (the case $\eta = .5$). Computations were carried out for a special case of varying twist,

$$\theta_\tau = \varepsilon \sin(kz) \tag{75}$$

at $z = 0$, $\pi/(4k)$, $\pi/(2k)$, etc. In this case, the profiles of δV_z are asymmetric. The complex nature of the results made it clear that the reader who wishes specific detail for any given $\theta_\tau(z)$ should construct the corresponding δV_z for himself, using equation (71) and Tables 1, 2 and 3.

6. Conclusions

The case of a (twisted) pipe of elliptical cross-section, though analytically tractable *with the aid of a computer*, is not completely representative of the general case. Also, most other special cases will require numerical solution of the governing partial differential equations and boundary conditions. For further conclusions, the reader is referred back to the comments in Section 4 (beginning with the first new paragraph after equation (37)).

It is a pleasure to record my gratitude to the National Research Council of Canada whose grant to me was of great benefit to the pursuit of my research. Mr. D. Wright was a very efficient programming assistant.

TABLE 2

Some numerical values of certain coefficients (see caption for Table 1)

η	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
e_{11}	−7	−6	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2
	−.226	−.707	−.501	−.188	−.493	−.102	−.178	−.276	−.391	−.515	−.643	−.770	−.895	−.102	−.113	−.125	−.137	−.149	−.161	−.174
e_{13}	−7	−5	−5	−4	−4	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.421	.132	.942	.356	.940	.196	.347	.544	.778	.104	.132	.160	.189	.218	.248	.278	.310	.343	.379	.417
e_{31}	−6	−5	−4	−4	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.105	.323	.222	.802	.200	.393	.652	.961	.130	.164	.196	.227	.255	.281	.305	.327	.349	.371	.394	.417
e_{33}	−6	−5	−4	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	−.154	−.474	−.327	−.118	−.296	−.583	−.971	−.144	−.195	−.248	−.300	−.350	−.398	−.444	−.489	−.534	−.579	−.626	−.676	−.729
e_{15}	−7	−6	−5	−4	−4	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2
	−.261	−.824	−.590	−.225	−.599	−.126	−.226	−.360	−.525	−.714	−.924	−.115	−.139	−.164	−.191	−.220	−.251	−.286	−.323	−.365
e_{51}	−6	−5	−4	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	−.180	−.544	−.365	−.127	−.305	−.571	−.903	−.127	−.163	−.197	−.227	−.253	−.274	−.292	−.307	−.319	−.331	−.342	−.353	−.365
e_{55}	−7	−5	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−2
	−.428	−.131	−.887	−.313	−.755	−.141	−.223	−.310	−.395	−.474	−.543	−.603	−.655	−.703	−.749	−.796	−.847	−.904	−.968	−.104
e_{17}	−8	−6	−5	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2
	.680	.214	.153	.582	.155	.327	.590	.949	.140	.195	.258	.330	.412	.504	.608	.727	.863	.102	.119	.139
e_{71}	−6	−5	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.135	.401	.263	.891	.206	.370	.561	.753	.928	.108	.119	.127	.133	.137	.138	.139	.139	.139	.139	.139
e_{19}	−9	−8	−7	−6	−6	−6	−5	−5	−5	−5	−5	−4	−4	−4	−4	−4	−4	−3	−3	−3
	−.222	−.657	−.424	−.141	−.326	−.618	−.108	−.186	−.321	−.547	−.905	−.144	−.219	−.321	−.453	−.622	−.831	−.108	−.138	−.174
e_{91}	−7	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	−.373	−.110	−.706	−.233	−.521	−.903	−.132	−.170	−.201	−.223	−.237	−.243	−.243	−.239	−.231	−.221	−.210	−.198	−.186	−.174
e_{35}	−7	−5	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.690	.213	.148	.540	.136	.268	.448	.665	.906	.116	.142	.168	.194	.221	.248	.276	.307	.340	.337	.417
e_{53}	−6	−5	−4	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.181	.550	.369	.129	.310	.581	.919	.129	.166	.201	.233	.261	.285	.306	.325	.343	.360	.378	.397	.417
e_{37}	−7	−6	−5	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	−.108	−.333	−.229	−.822	−.202	−.387	−.624	−.894	−.118	−.148	−.178	−.210	−.245	−.285	−.330	−.382	−.444	−.516	−.599	−.694
e_{73}	−7	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	−.695	−.208	−.137	−.465	−.108	−.194	−.293	−.392	−.481	−.554	−.610	−.650	−.675	−.689	−.696	−.697	−.696	−.694	−.693	−.694

TABLE 3

Some numerical values of certain coefficients (see caption for Table 1)

η	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95	1.00
e_{00}	−8	−6	−6	−5	−5	−5	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4	−18
	−.998	−.150	−.688	−.192	−.408	−.727	−.115	−.167	−.225	−.286	−.344	−.394	−.429	−.444	−.434	−.398	−.334	−.244	−.131	−.203
e_{10}	−6	−5	−5	−4	−4	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	.127	.165	.628	.147	.265	.414	.583	.765	.952	.113	.130	.145	.158	.168	.175	.179	.181	.180	.178	.174
e_{01}	−9	−7	−7	−6	−6	−5	−5	−5	−5	−5	−4	−4	−4	−4	−4	−4	−4	−3	−3	−3
	−.693	−.116	−.607	−.198	−.450	−.107	−.205	−.361	−.600	−.949	−.144	−.212	−.301	−.416	−.560	−.734	−.940	−.118	−.144	−.174
e_{20}	−6	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2
	.173	.256	.115	.313	.655	.117	.186	.274	.382	.507	.646	.796	.952	.111	.125	.139	.151	.161	.168	.174
e_{40}	−6	−5	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	−.586	−.840	−.361	−.945	−.190	−.327	−.506	−.727	−.989	−.129	−.161	−.196	−.232	−.268	−.302	−.333	−.361	−.384	−.403	−.417
e_{04}	−7	−6	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2
	.261	.440	.234	.775	.198	.426	.816	.143	.235	.365	.540	.766	.105	.138	.177	.220	.266	.315	.366	.417
e_{60}	−6	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.819	.114	.468	.118	.228	.378	.567	.790	.104	.132	.162	.192	.223	.252	.280	.304	.325	.342	.355	.365
e_{06}	−7	−6	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2
	−.229	−.383	−.202	−.663	−.168	−.359	−.686	−.120	−.197	−.305	−.451	−.640	−.877	−.116	−.149	−.187	−.228	−.272	−.318	−.365
e_{80}	−6	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2
	−.523	−.702	−.278	−.673	−.126	−.202	−.293	−.397	−.509	−.627	−.746	−.863	−.975	−.108	−.117	−.124	−.130	−.134	−.137	−.139
e_{08}	−8	−6	−6	−5	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2
	.780	.130	.687	.225	.569	.122	.233	.408	.670	.104	.155	.222	.306	.410	.533	.674	.833	.101	.119	.139
e_{22}	−6	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−4	−16
	−.147	−.211	−.897	−.229	−.444	−.723	−.105	−.140	−.176	−.209	−.239	−.262	−.275	−.277	−.266	−.240	−.199	−.144	−.774	−.104
e_{42}	−6	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2
	.437	.610	.252	.630	.121	.199	.297	.415	.556	.724	.920	.115	.141	.170	.201	.234	.267	.301	.334	.365
e_{24}	−7	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2	−2
	−.632	−.109	−.584	−.192	−.478	−.991	−.180	−.297	−.455	−.654	−.893	−.117	−.147	−.180	−.213	−.246	−.279	−.310	−.338	−.365
e_{62}	−6	−5	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2
	−.431	−.579	−.229	−.552	−.103	−.165	−.240	−.329	−.433	−.556	−.699	−.865	−.106	−.127	−.150	−.175	−.201	−.227	−.253	−.278
e_{26}	−7	−5	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−2	−2	−2	−2	−2	−2	−2	−2
	.643	.106	.546	.173	.419	.850	.152	.248	.375	.535	.725	.940	.118	.142	.168	.193	.217	.239	.259	.278
e_{82}	−6	−5	−5	−4	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	.141	.183	.694	.161	.289	.447	.628	.827	.104	.127	.152	.180	.211	.245	.284	.326	.372	.421	.470	.521
e_{28}	−7	−6	−5	−5	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	−.155	−.255	−.131	−.413	−.993	−.200	−.354	−.569	−.850	−.119	−.159	−.202	−.248	−.294	−.339	−.382	−.422	−.459	−.491	−.521
e_{44}	−7	−6	−5	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−16
	.491	.900	.509	.171	.427	.871	.154	.243	.350	.469	.587	.692	.770	.809	.800	.738	.622	.456	.246	.112
e_{64}	−7	−6	−5	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−4	−4	−3	−3
	−.120	−.253	−.157	−.554	−.141	−.292	−.517	−.815	−.117	−.153	−.187	−.211	−.220	−.210	−.175	−.115	−.299	−.781	−.205	−.347
e_{46}	−7	−6	−5	−4	−4	−4	−4	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
	−.414	−.680	−.348	−.109	−.259	−.515	−.896	−.141	−.204	−.277	−.353	−.427	−.491	−.540	−.568	−.572	−.550	−.503	−.435	−.347

REFERENCES

[1] G. K. Batchelor, *An Introduction to Fluid Mechanics*, Cambridge Univ. Press (1967).
[2] M. J. Manton, Long-wavelength peristaltic pumping at low Reynolds numbers, *J. Fluid Mech.* 68 (1975) 467–476.
[3] S. V. Patankar, V. S. Pratap and D. B. Spalding, Prediction of turbulent flow in curved pipes, *J. Fluid Mech.* 67 (1975) 583–595.
[4] W. M. Collins and S. C. R. Dennis, The steady motion of a viscous fluid in a curved tube, *Quart. J. Mech. and Appl. Math.* 28 (1975) 133–156.
[5] P. Hall, Unsteady viscous flow in a pipe of slowly varying cross-section, *J. Fluid Mech.* 64 (1974) 209–226.
[6] H. A. Barnes and K. Walters, On the flow of viscous and elastico-viscous liquids through straight and curved pipes, *Proc. Roy. Soc.* A314 (1969) 85–109.
[7] M. R. Davidson and J. M. Fitz-Gerald, Flow patterns in models of small airway units of the lung, *J. Fluid Mech.* 52 (1972) 161–177.
[8] M. J. Lighthill, *Mathematical Biofluiddynamics*, S.I.A.M. Press (1975) §§ 10–14.
[9] M. J. Manton, Low Reynolds number flow in slowly varying axisymmetric tubes, *J. Fluid Mech.* 49 (1971) 451–459.
[10] W. R. Dean, Note on the motion of fluid in a curved pipe, *Phil. Mag.* 4 (1927) 208–223.
[11] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications (1944).
[12] S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill (1959).
[13] I. A. Wojtaszak, The calculation of maximum deflection, moment, and shear for uniformly loaded rectangular plate with clamped edges, *J. Appl. Mech.* 4 (1937) 173–176.